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Extremal results on disjoint cycles in tournaments and bipartite tournaments. (English)

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All graphs considered in this paper are simple and directed. Denote the minimum out-degree of a digraph D by $\delta^+(D)$. A tournament is an orientation of the edges of a complete graph, and a bipartite tournament is an orientation of the edges of a complete bipartite graph. By a $(\geq q)$-cycle we mean a directed cycle of length at least q, $q \geq 2$. In this work, the author considers the problem of finding collections of pairwise vertex-disjoint directed cycles (briefly, disjoint cycles) in digraphs under specific conditions, which falls under a long and fruitful line of investigation initiated by *J. C. Bermond* and *C. Thomassen* [J. Graph Theory 5, 1–43 (1981; Zbl 0458.05035)].

N. D. Tan [SIAM J. Discrete Math. 35, No. 1, 485–494 (2021; Zbl 1460.05077)] and *B. Chen* and *A. Chang* [J. Graph Theory 105, No. 2, 297–314 (2024; Zbl 1531.05100)] proved that, for any $k \geq 2$, any strongly connected tournament T with $\delta^+(T) \geq 2k-1$ contains a set of k disjoint cycles such that two of them have distinct lengths, except when T is isomorphic to the Paley tournament on 7 vertices. They also proved that, for any $k \geq 2$, any strongly connected bipartite tournament T with $\delta^+(T) \geq 2k-1$ contains a set of k disjoint cycles such that two of them have distinct lengths, except when T is isomorphic to $\mathrm{BT}(n_1, \dots, n_{2k})$ (for some choices of $n_i \geq 2k-1$), which is defined as follows. Given integers $k \geq 2$, $q \geq 2$, and $n_i \geq qk-1$ for $1 \leq i \leq qk$, the bipartite tournament $\mathrm{BT}(n_1, \dots, n_{qk})$ has bipartition $V = X \cup Y$, where X is a disjoint union of sets $X_1, \dots, X_{qk}$ with $|X_i| = n_i$ for $1 \leq i \leq qk$, and $Y = \{y_1, \dots, y_{qk}\}$ is a set of size qk. Given $x \in X$ and $y_j \in Y$, the arc between these vertices is directed as $x \to y_j$ if $x \in X_i$ with $i \neq j$, and as $y_j \to x$ if $x \in X_j$.

In this paper, the author extends these results by proving the following two results. First, for any $k \geq 2$ and $q \geq 3$, any strongly connected tournament T with $\delta^+(T) \geq (q-1)k-1$ contains a set of k disjoint $(\geq q)$-cycles such that two of them have distinct lengths, except when T is isomorphic to the Paley tournament on 7 vertices. Second, for any $k \geq 2$ and even $q \geq 2$, any strongly connected bipartite tournament T with $\delta^+(T) \geq qk-1$ contains a set of k disjoint $(\geq 2q)$-cycles such that two of them have distinct lengths, except when T is isomorphic to $\mathrm{BT}(n_1, \dots, n_{qk})$ (for some choices of $n_i \geq qk-1$).

As a corollary, the author deduces similar extremal results for weakly connected tournaments and weakly connected bipartite tournaments. The author also raises the problem of characterizing the bipartite tournaments T with $\delta^+(T) \geq qk-1$ for odd q in which every collection $\mathcal{C}$ of k disjoint $(\geq 2q)$-cycles has the property that all the cycles in $\mathcal{C}$ are of the same length.

The proofs are by contradiction, but can easily be made direct with very few changes. In the proof of Claim 1, the path P_2 should be the shortest path ending at w_2 and starting at some vertex of U, instead of the other way around. Following the proof of Claim 2, one must observe that the out-neighbors of $V(C_1)$ all belong to $V(C_1) \cup V(C_2)$, and not just $V(C_2)$. The proof of Case 2 within the proof of Theorem 3.2 should show that $\kappa_5^2(D) = 2$, instead of $\kappa_4^2(D) = 2$; a simple change in the definition of C_1^* achieves this. In the proof of Theorem 3.3, the number of strongly connected components of $D[W]$ is taken to be h, but later on in the proof, h is reused as a dummy index varying over $[t-1]$. Some minor typos can be easily corrected based on the context.

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05C20 Directed graphs (digraphs), tournaments
05C38 Paths and cycles
05C35 Extremal problems in graph theory

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